THE
# MATHEMATICAL GAZETTE.

EDITED BY
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WITH THE CO-OPERATION OF
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## REPORT ON THE TEACHING OF THE MULTIPLICATION AND DIVISION OF DECIMALS.

THE General Teaching Committee of the Mathematical Association has authorised the publication of the following investigation by the Public Schools Special Committee. The General Committee does not commit itself to any expression of opinion as to the comparative merits of various methods for the multiplication and division of decimals; the methods given are those which seemed to the Special Committee the only ones likely to secure general support.

The Special Committee has also been authorised to take such steps as seem most suitable to make the result of the investigation known to Preparatory Schools.

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In July, 1914, the Public Schools Special Committee sent out to the schools mentioned in the *Public Schools' Year Book* (1914) the Questionnaire and accompanying letter given below. Of these schools 83 sent replies. The result of the enquiry is as follows:

Qu. 2. Do you think it desirable that all boys coming to the School should be familiar with one uniform method for

(*a*) multiplication of decimals?
(*b*) division of decimals?

Of the 83 schools 76 are in favour of uniformity, 1 is definitely against, 2 do not answer definitely but appear to be against, 2 do not answer definitely but are decidedly favourable, and 2 are quite indefinite.

Qu. 3. If you think one uniform method desirable, would you like it to be the one set out below for

(*a*) multiplication? (*b*) division?

(3*a*). 65 schools are in favour of the given method for multiplication, 9 are opposed to it, 3 do not answer quite definitely but are on the whole favourable, the rest do not answer definitely. [Schools were not asked to choose between the alternative methods of arrangement, but, of the 64 schools in favour of the given method, 19 specified (*b*) and 5 specified (*a*).]

(3*b*). 65 schools are in favour of the given method for division, 8 are opposed to it, 3 do not answer quite definitely but are on the whole favourable, the rest do not answer definitely.

Qu. 4. If you approve of the method indicated, may the name of your School be given as so approving, if a circular is issued to the Preparatory Schools ?

64 schools are willing that their names should be given as approving, 1 senior mathematical master approves of the methods but has not power to speak for the school.

Qu. 5. If you prefer some other method—but 70 per cent. of the Public Schools replying to this questionnaire declare in favour of the method given below—are you willing for the sake of uniformity to allow the name of your school to be given in the circular referred to in Question 4 above ?

11 schools are willing to allow their names to appear, 4 are not willing.

As the 70 per cent. has been obtained, this means that altogether 75 schools are willing that their names should appear in a circular to the Preparatory Schools, 4 are definitely opposed to their names appearing.

The following list gives the names of the 75 schools (out of the 82 that replied) that are willing that their names should appear :

Aldenham School.
Downside School, Bath.
Bedford Grammar School.
Campbell College, Belfast.
Berkhamsted School.
Bishop's Stortford.
Bradfield College.
Christ College, Brecon.
Brighton College.
Bristol Grammar School.
Bromsgrove School.
Leys School, Cambridge.
St. Edmund's School, Canterbury.
The Charterhouse.
Cheltenham College.
Dean Close School, Cheltenham.
The King's School, Chester.
Chigwell School.
Christ's Hospital.
City of London School.
Clifton College.
Whitgift Grammar School, Croydon.
Denstone College.
Dover College.
Durham School.
Eastbourne College.
Epsom College.
Exeter School.
Fettes College.
Giggleswick School.
Trinity College, Glenalmond.
Haileybury College.
Harrow School.
Highgate School.
Gresham's School, Holt, Norfolk.
Hymer's College, Hull.
King William's College, Isle of Man.
Victoria College, Jersey.
Lancaster Royal Grammar School.
Lancing College.
Leeds Grammar School.
Loretto School.
Magdalen College School.
Malvern College.
Marlborough College.
Merchant Taylor's School, E.C.
Merchiston Castle School, Edinburgh.
Mill Hill School.
Monkton Combe School, Bath.
Nottingham High School.
Oakham School.
Royal Naval College, Osborne.
Oundle School.
St. Edward's School, Oxford.
Plymouth College.
Portsmouth Grammar School.
Reading School.
Repton School.
Rossall School.
Rugby School.
St. Alban's School.
Sedbergh School.
Sherborne School.
Shrewsbury School.
St. Olave's School, Southwark.
Stonyhurst College.
Blundell's School, Tiverton.
Tonbridge School.
Wakefield Grammar School.
Warwick School.
Wellington College.
Beaumont College, Windsor.
Winchester College.
Royal Grammar School, Worcester.
St. Peter's School, York.

## QUESTIONNAIRE.

1. Name of School :

2. Do you think it desirable that all boys coming to the School should be familiar with one uniform method for

(*a*) multiplication of decimals ? (*b*) division of decimals ?

3. If you think one uniform method desirable, would you like that method to be the one set out below for

(*a*) multiplication ? (*b*) division ?

4. If you approve of the method indicated, may the name of your School be given as so approving, if a circular is issued to the Preparatory Schools ?

5. If you prefer some other method—but 70 per cent. of the Public Schools replying to this questionnaire declare in favour of the method given below—are you willing for the sake of uniformity to allow the name of your school to be given in the circular referred to in Question 4 above ?

6. If you prefer some other method, will you indicate that method ?

The methods suggested are the following :

### Multiplication.

To multiply 17·63 by 34·1.

Reduce the multiplier to standard form :

$$17{\cdot}63 \times 34{\cdot}1 = 176{\cdot}3 \times 3{\cdot}41.$$

The process can then be set out in *either of the following ways* :

(*a*)
$$\begin{array}{l}
176{\cdot}3 \\
\underline{\phantom{17}3{\cdot}41} \\
528{\cdot}9 \\
\phantom{5}70{\cdot}52 \\
\underline{\phantom{52}1{\cdot}763} \\
601{\cdot}183
\end{array}$$

(*b*)
$$\begin{array}{l}
176{\cdot}3 \\
\underline{\phantom{17}3{\cdot}41} \\
528{\cdot}9 \\
\phantom{5}70{\cdot}52 \\
\underline{\phantom{52}1{\cdot}763} \\
601{\cdot}183
\end{array}$$

In (*a*) the unit figure of the multiplier is under the right hand figure of the multiplicand, and the decimal point in the answer under the decimal point in the multiplicand.

In (*b*) the unit figure of the multiplier is under the unit figure of the multiplicand, and all decimal points are in a vertical line.

### Division.

To divide 17·63 by 34·1. Reduce the divisor to standard form :

$$\frac{17{\cdot}63}{34{\cdot}1} = \frac{1{\cdot}763}{3{\cdot}41}.$$

$$\begin{array}{r|l}
 & \phantom{1}{\cdot}517\ldots \\
3{\cdot}41 & 1{\cdot}763 \\
 & \underline{1{\cdot}705\phantom{0}} \\
 & \phantom{1{\cdot}}580 \\
 & \phantom{1{\cdot}}\underline{341} \\
 & \phantom{1{\cdot}}2390
\end{array}$$

The position of the first figure of the quotient is found by inspection as in short division.

## LETTER SENT WITH THE QUESTIONNAIRE.

Christ's Hospital,

*July* 13*th*, 1914.

Dear Sir,

I have been instructed by the Public Schools Special Committee of the Mathematical Association to send the enclosed circular to the Senior Mathematical Masters of Public Schools.

The Committee, though fully recognising the desirability of variety of treatment in the teaching of Mathematics as a general rule, is of opinion that uniformity of method in certain fundamental processes in Arithmetic would be a great convenience, especially in multiplication and division of decimal fractions.

If boys remained at the same school during the whole of their Arithmetic course, each school would be independent in this matter and might reasonably object to any attempt to influence its actions. But such a large proportion of boys at some time change their school (principally in passing from a Preparatory School to a Public School), that the work in certain parts of the Public Schools is handicapped by the want of uniformity of method.

The Committee is consequently endeavouring to find out whether such uniformity is generally considered desirable as far as boys entering the Public Schools are concerned, and, if so, whether anything can be done to secure, if not complete uniformity, at any rate some approximation to it.

At a preliminary discussion the Committee examined the different methods in common use. It appeared that, though much might be said in favour of various methods, the only one at all likely to secure general support is the one described in the accompanying circular.

If the replies to these questions should lead to any result, it is proposed to inform the Preparatory Schools, and it is hoped that a considerable number of them will adopt the methods agreed upon.—Yours faithfully,

G. W. PALMER, *Hon. Sec.*

## THE BORDERED ANTILOGARITHM TABLE.

BY PROF. G. H. BRYAN, F.R.S., AND T. G. CREAK, M.A.

WHILE in recent years the market has been flooded with small books of mathematical tables, each of which reproduces most of the defects of the others, there are few tables, if any, which meet all requirements when it is necessary to use logarithms for operations involving both multiplication and division. In such cases logarithms of reciprocals are constantly needed. Few tables contain these, a notable exception being the tables at the end of Carson and Smith's *Algebra*. On the other hand, tables of reciprocals are often given, which are rarely of use except in geometrical optics, and only confusion is occasioned by the separation of the tables for logarithms and antilogarithms, and of those for sines and cosines. Either the table of antilogarithms or that of logarithms may be banished with advantage.

If we retain the table of logarithms, it is impossible to obtain logarithms of reciprocals without performing an additional operation, which increases the work and the risk of error. Further, in the lowest parts of the scale the differences are large, and it is difficult to pick out the number which represents the logarithm *correctly* to four or five places of decimals as the case may be.

Now a table of *antilogarithms* may be used to find logarithms of reciprocals in just the same way that sines and cosines can be taken from the same table. The left-hand column and top line contain the actual logarithms of the numbers tabulated in the area of the present table, while the right-hand column and bottom line give the complementary logarithms or logarithms of the reciprocals.

Moreover, in order to secure greater uniformity in the degree of accuracy of working, it will be seen that on the first page the results are tabulated to five significant figures, and on the second to four. The reason of this is that, in order to find the logarithm of a number

correct to four places, the number must be given to five figures in the lower parts of the scale, whereas in the higher parts it need only be given to four figures, and, furthermore, a number in the higher parts of the scale can only be found correct to four figures from a table of four-figure logarithms. It will be seen that by this method the mean differences are neither too small nor too large. The important point to be noticed is that, in order to find a logarithm correct to four places, the mean differences must not be the same in any two columns, and, to ensure this, numbers less than 4 must be known correctly to five significant figures. Two examples will make this clear.

*Ex.* 1. Multiply 1·058 by 8·926, using four-figure logarithms, the data being approximate.

Here 1·058 means some number between 1·0575 and $1{\cdot}0584\dot{9}$. With the present table we find, therefore,

| | | | | |
|---|---|---|---|---|
| log 1·058 | between | ·0243 | and | ·0247 |
| log 8·926 | = | ·9506 | = | ·9506 |
| log product | between | ·9749 | and | ·9753 |
| product | between | 9·439 | and | 9·448 |

*Ex.* 2. Find the number whose four-figure logarithm is ·9457.

Even if the result has *not* been arrived at by the addition of several logarithms, it represents a logarithm lying between ·94565 and $\cdot 94574\dot{9}$. If we use a table of five-figure antilogarithms, or a seven-figure table, we shall find that the number lies between 8·8237 and 8·8257. Thus, even if we take the result as 8·825 to four significant figures, there is a possible error of $\pm 1$ in the last place, and the fifth significant figure is of no value whatever.

More generally, since a four-figure logarithm represents an approximate value of a logarithm, the true value of which may differ from it by $\pm 0{\cdot}00005$, and since $0{\cdot}00005 = \log 1{\cdot}000115$, it follows that the number, when determined from a four-figure logarithm, may differ from its correct value by $\pm 0{\cdot}000115$ of the whole, or 0·0115 per cent. This does not take into account the cumulative error introduced when logarithms are added together or multiplied by some factor in the calculation of powers.

We may look at the matter in another way by regarding four-figure logarithms as a collection of 10,000 numbers from 0000 to 9999. Accuracy in working depends on choosing the correct one of these numbers, and this is facilitated by tabulating each one separately, as is done here, and giving the required data to the degree of approximation necessary to make the correct choice in every instance.

As the second half of the table does not occupy a whole page, the space is filled up by tables of the logarithms and logarithms of reciprocals of numbers from 10 to 99, as well as the logs and log reciprocals of a few of the most important constants. As there is room on the page, these are given to five places, a plan which should secure greater accuracy when powers and products are found, owing to figures carrying from the fifth to the fourth place.

Personally I have always held that logarithms should be taught before indices, and if this is done it may be necessary to defer the introduction of this table till the pupils have got fairly advanced in their work. As, however, the opposite practice commonly prevails, the use of these tables should, in most cases, facilitate matters. For this reason I have headed the tables "Antilogarithms or Powers of 10." It may be found even better to drop the somewhat clumsy name "Antilogarithm" altogether, and to call the table merely one of "Powers of 10." Whether this change is desirable is probably a point on which differences of opinion exist.

ANTILOGARITHMS.

| Index or Log. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Differences. 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ·00 | 10000 | 023 | 046 | 069 | 093 | 116 | 139 | 162 | 186 | 209 | 10233 | 2 | 5 | 7 | 9 | 12 | 14 | 16 | 19 | 21 | ·99 |
| ·01 | 10233 | 257 | 280 | 304 | 328 | 351 | 375 | 399 | 423 | 447 | 10471 | 2 | 5 | 7 | 10 | 12 | 14 | 17 | 19 | 21 | ·98 |
| ·02 | 10471 | 495 | 520 | 544 | 568 | 593 | 617 | 641 | 666 | 691 | 10715 | 2 | 5 | 7 | 10 | 12 | 15 | 17 | 20 | 22 | ·97 |
| ·03 | 10715 | 740 | 765 | 789 | 814 | 839 | 864 | 889 | 914 | 940 | 10965 | 3 | 5 | 8 | 10 | 13 | 15 | 18 | 20 | 23 | ·96 |
| ·04 | 10965 | 990 | 015 | 041 | 066 | 092 | 117 | 143 | 169 | 194 | 11220 | 3 | 5 | 8 | 10 | 13 | 15 | 18 | 20 | 23 | ·95 |
| ·05 | 11220 | 246 | 272 | 298 | 324 | 350 | 376 | 402 | 429 | 455 | 11482 | 3 | 5 | 8 | 11 | 13 | 16 | 18 | 21 | 24 | ·94 |
| ·06 | 11482 | 508 | 535 | 561 | 588 | 614 | 641 | 668 | 695 | 722 | 11749 | 3 | 5 | 8 | 11 | 13 | 16 | 19 | 21 | 24 | ·93 |
| ·07 | 11749 | 776 | 803 | 830 | 858 | 885 | 912 | 940 | 967 | 995 | 12023 | 3 | 5 | 8 | 11 | 14 | 16 | 19 | 22 | 25 | ·92 |
| ·08 | 12023 | 050 | 078 | 106 | 134 | 162 | 190 | 218 | 246 | 274 | 12303 | 3 | 6 | 8 | 11 | 14 | 17 | 20 | 22 | 25 | ·91 |
| ·09 | 12303 | 331 | 359 | 388 | 417 | 445 | 474 | 503 | 531 | 560 | 12589 | 3 | 6 | 9 | 11 | 14 | 17 | 20 | 23 | 26 | ·90 |
| ·10 | 12589 | 618 | 647 | 677 | 706 | 735 | 764 | 794 | 823 | 853 | 12882 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 26 | ·89 |
| ·11 | 12882 | 912 | 942 | 972 | 002 | 032 | 062 | 092 | 122 | 152 | 13183 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | ·88 |
| ·12 | 13183 | 213 | 243 | 274 | 305 | 335 | 366 | 397 | 428 | 459 | 13490 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 25 | 28 | ·87 |
| ·13 | 13490 | 521 | 552 | 583 | 614 | 646 | 677 | 709 | 740 | 772 | 13804 | 3 | 6 | 9 | 13 | 16 | 19 | 22 | 25 | 28 | ·86 |
| ·14 | 13804 | 836 | 868 | 900 | 932 | 964 | 996 | 028 | 060 | 093 | 14125 | 3 | 6 | 10 | 13 | 16 | 19 | 22 | 26 | 29 | ·85 |
| ·15 | 14125 | 158 | 191 | 223 | 256 | 289 | 322 | 355 | 388 | 421 | 14454 | 3 | 7 | 10 | 13 | 16 | 20 | 23 | 26 | 30 | ·84 |
| ·16 | 14454 | 488 | 521 | 555 | 588 | 622 | 655 | 689 | 723 | 757 | 14791 | 3 | 7 | 10 | 13 | 17 | 20 | 24 | 27 | 30 | ·83 |
| ·17 | 14791 | 825 | 859 | 894 | 928 | 962 | 997 | 031 | 066 | 101 | 15136 | 3 | 7 | 10 | 14 | 17 | 21 | 24 | 28 | 31 | ·82 |
| ·18 | 15136 | 171 | 205 | 241 | 276 | 311 | 346 | 382 | 417 | 453 | 15488 | 4 | 7 | 11 | 14 | 18 | 21 | 25 | 28 | 32 | ·81 |
| ·19 | 15488 | 524 | 560 | 596 | 631 | 668 | 704 | 740 | 776 | 812 | 15849 | 4 | 7 | 11 | 14 | 18 | 22 | 25 | 29 | 32 | ·80 |
| ·20 | 15849 | 885 | 922 | 959 | 996 | 032 | 069 | 106 | 144 | 181 | 16218 | 4 | 7 | 11 | 15 | 18 | 22 | 26 | 30 | 33 | ·79 |
| ·21 | 16218 | 255 | 293 | 331 | 368 | 406 | 444 | 482 | 520 | 558 | 16596 | 4 | 8 | 11 | 15 | 19 | 23 | 26 | 30 | 34 | ·78 |
| ·22 | 16596 | 634 | 672 | 711 | 749 | 788 | 827 | 866 | 904 | 943 | 16982 | 4 | 8 | 12 | 15 | 19 | 23 | 27 | 31 | 35 | ·77 |
| ·23 | 16982 | 022 | 061 | 100 | 140 | 179 | 219 | 258 | 298 | 333 | 17378 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | ·76 |
| ·24 | 17378 | 418 | 458 | 498 | 539 | 579 | 620 | 660 | 701 | 742 | 17783 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | ·75 |
| ·25 | 17783 | 824 | 865 | 906 | 947 | 989 | 030 | 072 | 113 | 155 | 18197 | 4 | 8 | 12 | 17 | 21 | 25 | 29 | 33 | 37 | ·74 |
| ·26 | 18197 | 239 | 281 | 323 | 365 | 408 | 450 | 493 | 535 | 578 | 18621 | 4 | 8 | 13 | 17 | 21 | 25 | 30 | 34 | 38 | ·73 |
| ·27 | 18621 | 664 | 707 | 750 | 793 | 836 | 880 | 923 | 967 | 011 | 19055 | 4 | 9 | 13 | 17 | 22 | 26 | 30 | 35 | 39 | ·72 |
| ·28 | 19055 | 099 | 143 | 187 | 231 | 275 | 320 | 364 | 409 | 454 | 19498 | 4 | 9 | 13 | 18 | 22 | 26 | 31 | 35 | 40 | ·71 |
| ·29 | 19498 | 543 | 588 | 634 | 679 | 724 | 770 | 815 | 861 | 907 | 19953 | 5 | 9 | 14 | 18 | 23 | 27 | 32 | 36 | 41 | ·70 |
| ·30 | 19953 | 999 | 045 | 091 | 137 | 184 | 230 | 277 | 324 | 370 | 20417 | 5 | 9 | 14 | 19 | 23 | 28 | 32 | 37 | 42 | ·69 |
| ·31 | 20417 | 464 | 512 | 559 | 606 | 654 | 701 | 749 | 797 | 845 | 20893 | 5 | 10 | 14 | 19 | 24 | 29 | 33 | 38 | 43 | ·68 |
| ·32 | 20893 | 941 | 989 | 038 | 086 | 135 | 184 | 232 | 281 | 330 | 21380 | 5 | 10 | 15 | 19 | 24 | 29 | 34 | 39 | 44 | ·67 |
| ·33 | 21380 | 429 | 478 | 528 | 577 | 627 | 677 | 727 | 777 | 827 | 21878 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | ·66 |
| ·34 | 21878 | 928 | 979 | 029 | 080 | 131 | 182 | 233 | 284 | 336 | 22387 | 5 | 10 | 15 | 29 | 25 | 31 | 36 | 41 | 46 | ·65 |
| ·35 | 22387 | 439 | 491 | 542 | 594 | 646 | 699 | 751 | 803 | 856 | 22909 | 5 | 10 | 16 | 21 | 26 | 31 | 37 | 42 | 47 | ·64 |
| ·36 | 22909 | 961 | 014 | 067 | 121 | 174 | 227 | 281 | 336 | 388 | 23442 | 5 | 11 | 16 | 21 | 27 | 32 | 37 | 43 | 48 | ·63 |
| ·37 | 23442 | 496 | 550 | 605 | 659 | 714 | 768 | 823 | 878 | 933 | 23988 | 5 | 11 | 16 | 22 | 27 | 33 | 38 | 44 | 49 | ·62 |
| ·38 | 23988 | 044 | 099 | 155 | 210 | 266 | 322 | 378 | 434 | 491 | 24547 | 6 | 11 | 17 | 22 | 28 | 34 | 39 | 45 | 50 | ·61 |
| ·39 | 24547 | 604 | 660 | 717 | 774 | 831 | 889 | 946 | 003 | 061 | 25119 | 6 | 11 | 17 | 23 | 29 | 34 | 40 | 46 | 51 | ·60 |
| ·40 | 25119 | 177 | 236 | 293 | 351 | 410 | 468 | 527 | 586 | 645 | 25704 | 6 | 12 | 18 | 23 | 29 | 35 | 41 | 47 | 53 | ·59 |
| ·41 | 25704 | 763 | 823 | 882 | 942 | 002 | 062 | 122 | 182 | 242 | 26303 | 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | ·58 |
| ·42 | 26303 | 363 | 424 | 485 | 546 | 607 | 669 | 730 | 792 | 853 | 26915 | 6 | 12 | 18 | 24 | 31 | 37 | 43 | 49 | 55 | ·57 |
| ·43 | 26915 | 977 | 040 | 102 | 164 | 227 | 290 | 353 | 416 | 479 | 27542 | 6 | 13 | 19 | 25 | 31 | 38 | 44 | 50 | 56 | ·56 |
| ·44 | 27542 | 606 | 669 | 733 | 797 | 861 | 925 | 990 | 054 | 119 | 28184 | 6 | 13 | 19 | 26 | 32 | 39 | 45 | 51 | 58 | ·55 |
| ·45 | 28184 | 249 | 314 | 379 | 445 | 510 | 576 | 642 | 708 | 774 | 28840 | 7 | 13 | 20 | 26 | 33 | 39 | 46 | 52 | 59 | ·54 |
| ·46 | 28840 | 907 | 973 | 040 | 107 | 174 | 242 | 309 | 376 | 444 | 29512 | 7 | 13 | 20 | 27 | 34 | 40 | 47 | 54 | 60 | ·53 |
| ·47 | 29512 | 580 | 648 | 717 | 785 | 854 | 923 | 992 | 061 | 130 | 30200 | 7 | 14 | 21 | 28 | 34 | 41 | 48 | 55 | 62 | ·52 |
| ·48 | 30200 | 259 | 339 | 409 | 479 | 549 | 620 | 690 | 761 | 832 | 30903 | 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | ·51 |
| ·49 | 30903 | 974 | 046 | 117 | 189 | 261 | 333 | 405 | 477 | 550 | 31623 | 7 | 14 | 22 | 29 | 36 | 43 | 50 | 58 | 65 | ·50 |
| ·50 | 31623 | 696 | 769 | 842 | 916 | 989 | 063 | 137 | 211 | 285 | 32359 | 7 | 15 | 22 | 29 | 37 | 44 | 52 | 59 | 66 | ·49 |
| ·51 | 32359 | 434 | 509 | 584 | 659 | 735 | 809 | 885 | 961 | 037 | 33113 | 8 | 15 | 23 | 30 | 38 | 45 | 53 | 60 | 68 | ·48 |
| ·52 | 33113 | 189 | 266 | 343 | 420 | 497 | 574 | 651 | 729 | 806 | 33884 | 8 | 15 | 23 | 31 | 39 | 46 | 54 | 62 | 69 | ·47 |
| ·53 | 33884 | 963 | 041 | 119 | 198 | 277 | 356 | 435 | 514 | 594 | 34674 | 8 | 16 | 24 | 32 | 40 | 47 | 55 | 63 | 71 | ·46 |
| ·54 | 34674 | 754 | 834 | 914 | 995 | 075 | 156 | 237 | 318 | 400 | 35481 | 8 | 16 | 24 | 32 | 40 | 48 | 56 | 65 | 73 | ·45 |
| ·55 | 35481 | 563 | 645 | 727 | 810 | 892 | 975 | 058 | 141 | 224 | 36308 | 8 | 16 | 25 | 33 | 41 | 50 | 58 | 66 | 74 | ·44 |
| ·56 | 36308 | 392 | 475 | 559 | 644 | 728 | 813 | 898 | 983 | 068 | 37154 | 8 | 17 | 25 | 34 | 42 | 51 | 59 | 68 | 76 | ·43 |
| ·57 | 37154 | 239 | 325 | 411 | 497 | 584 | 670 | 757 | 844 | 931 | 38019 | 9 | 17 | 26 | 35 | 43 | 52 | 61 | 69 | 78 | ·42 |
| ·58 | 38019 | 107 | 194 | 282 | 371 | 459 | 548 | 637 | 726 | 815 | 38905 | 9 | 18 | 27 | 35 | 44 | 53 | 62 | 71 | 80 | ·41 |
| ·59 | 38905 | 994 | 084 | 174 | 264 | 355 | 446 | 537 | 628 | 719 | 39811 | 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 82 | ·40 |
| ·60 | 39811 | 902 | 994 | 087 | 179 | 272 | 365 | 458 | 551 | 644 | 40738 | 9 | 19 | 28 | 37 | 46 | 56 | 65 | 74 | 83 | ·39 |
| | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | -1 | -2 | -3 | -4 | -5 | -6 | -7 | -8 | -9 | Colog |
| | | | | | | | | | | | | Differences. | | | | | | | | | |

ANTILOGARITHMIC RECIPROCALS.

ANTILOGARITHMS.

| Index or Log. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Differences. 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ·60 | 3981 | 3990 | 3999 | 4009 | 4018 | 4027 | 4036 | 4046 | 4055 | 4064 | 4074 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7 | 8 | ·39 |
| ·61 | 4074 | 4083 | 4093 | 4102 | 4111 | 4121 | 4130 | 4140 | 4150 | 4159 | 4169 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ·38 |
| ·62 | 4169 | 4178 | 4188 | 4198 | 4207 | 4217 | 4227 | 4236 | 4246 | 4256 | 4266 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ·37 |
| ·63 | 4266 | 4276 | 4285 | 4295 | 4305 | 4315 | 4325 | 4335 | 4345 | 4355 | 4365 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ·36 |
| ·64 | 4365 | 4375 | 4385 | 4395 | 4406 | 4416 | 4426 | 4436 | 4446 | 4457 | 4467 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ·35 |
| ·65 | 4467 | 4477 | 4487 | 4498 | 4508 | 4519 | 4529 | 4539 | 4550 | 4560 | 4571 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ·34 |
| ·66 | 4571 | 4581 | 4592 | 4603 | 4613 | 4624 | 4634 | 4645 | 4656 | 4667 | 4677 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 10 | ·33 |
| ·67 | 4677 | 4688 | 4699 | 4710 | 4721 | 4732 | 4742 | 4753 | 4764 | 4775 | 4786 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 | ·32 |
| ·68 | 4786 | 4797 | 4808 | 4819 | 4831 | 4842 | 4853 | 4864 | 4875 | 4887 | 4898 | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 9 | 10 | ·31 |
| ·69 | 4898 | 4909 | 4920 | 4932 | 4943 | 4955 | 4966 | 4977 | 4989 | 5000 | 5012 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 | ·30 |
| ·70 | 5012 | 5023 | 5035 | 5047 | 5058 | 5070 | 5082 | 5093 | 5105 | 5117 | 5129 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 9 | 11 | ·29 |
| ·71 | 5129 | 5140 | 5152 | 5164 | 5176 | 5188 | 5200 | 5212 | 5224 | 5236 | 5248 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 | ·28 |
| ·72 | 5248 | 5260 | 5272 | 5284 | 5297 | 5309 | 5321 | 5333 | 5346 | 5358 | 5370 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 | ·27 |
| ·73 | 5370 | 5383 | 5395 | 5408 | 5420 | 5433 | 5445 | 5458 | 5470 | 5483 | 5495 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 | ·26 |
| ·74 | 5495 | 5508 | 5521 | 5534 | 5546 | 5559 | 5572 | 5585 | 5598 | 5610 | 5623 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 | ·25 |
| ·75 | 5623 | 5636 | 5649 | 5662 | 5675 | 5689 | 5702 | 5715 | 5728 | 5741 | 5754 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 10 | 12 | ·24 |
| ·76 | 5754 | 5768 | 5781 | 5794 | 5808 | 5821 | 5834 | 5848 | 5861 | 5875 | 5888 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 | ·23 |
| ·77 | 5888 | 5902 | 5916 | 5929 | 5943 | 5957 | 5970 | 5984 | 5998 | 6012 | 6026 | 1 | 3 | 4 | 5 | 7 | 8 | 10 | 11 | 12 | ·22 |
| ·78 | 6026 | 6039 | 6053 | 6067 | 6081 | 6095 | 6109 | 6124 | 6138 | 6152 | 6166 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 13 | ·21 |
| ·79 | 6166 | 6180 | 6194 | 6209 | 6223 | 6237 | 6252 | 6266 | 6281 | 6295 | 6310 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 | ·20 |
| ·80 | 6310 | 6324 | 6339 | 6353 | 6368 | 6383 | 6397 | 6412 | 6427 | 6442 | 6457 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 | ·19 |
| ·81 | 6457 | 6471 | 6486 | 6501 | 6516 | 6531 | 6546 | 6561 | 6577 | 6592 | 6607 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 | ·18 |
| ·82 | 6607 | 6622 | 6637 | 6653 | 6668 | 6683 | 6699 | 6714 | 6730 | 6745 | 6761 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 | ·17 |
| ·83 | 6761 | 6776 | 6792 | 6808 | 6823 | 6839 | 6855 | 6871 | 6887 | 6902 | 6918 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 13 | 14 | ·16 |
| ·84 | 6918 | 6934 | 6950 | 6966 | 6982 | 6998 | 7015 | 7031 | 7047 | 7063 | 7079 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 13 | 15 | ·15 |
| ·85 | 7079 | 7096 | 7112 | 7129 | 7145 | 7161 | 7178 | 7194 | 7211 | 7228 | 7244 | 2 | 3 | 5 | 7 | 8 | 19 | 12 | 13 | 15 | ·14 |
| ·86 | 7244 | 7261 | 7278 | 7295 | 7311 | 7328 | 7345 | 7362 | 7379 | 7396 | 7413 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 15 | ·13 |
| ·87 | 7413 | 7430 | 7447 | 7464 | 7482 | 7499 | 7516 | 7534 | 7551 | 7568 | 7586 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 16 | ·12 |
| ·88 | 7586 | 7603 | 7621 | 7638 | 7656 | 7674 | 7691 | 7709 | 7727 | 7745 | 7762 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 | ·11 |
| ·89 | 7762 | 7780 | 7798 | 7816 | 7834 | 7852 | 7870 | 7889 | 7907 | 7925 | 7943 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 15 | 16 | ·10 |
| ·90 | 7943 | 7962 | 7980 | 7998 | 8017 | 8035 | 8054 | 8072 | 8091 | 8110 | 8128 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 | ·09 |
| ·91 | 8128 | 8147 | 8166 | 8185 | 8204 | 8222 | 8241 | 8260 | 8279 | 8299 | 8318 | 2 | 4 | 6 | 8 | 10 | 11 | 13 | 15 | 17 | ·08 |
| ·92 | 8318 | 8337 | 8356 | 8375 | 8395 | 8414 | 8433 | 8453 | 8472 | 8492 | 8511 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 | ·07 |
| ·93 | 8511 | 8531 | 8551 | 8570 | 8590 | 8610 | 8630 | 8650 | 8670 | 8690 | 8710 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | ·06 |
| ·94 | 8710 | 8730 | 8750 | 8770 | 8790 | 8810 | 8831 | 8851 | 8872 | 8892 | 8913 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | ·05 |
| ·95 | 8913 | 8933 | 8954 | 8974 | 8995 | 9016 | 9036 | 9057 | 9078 | 9099 | 9120 | 2 | 4 | 6 | 8 | 10 | 12 | 15 | 17 | 19 | ·04 |
| ·96 | 9120 | 9141 | 9162 | 9183 | 9204 | 9226 | 9247 | 9268 | 9290 | 9311 | 9333 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 | ·03 |
| ·97 | 9333 | 9354 | 9376 | 9397 | 9419 | 9441 | 9462 | 9484 | 9506 | 9528 | 9550 | 2 | 4 | 7 | 9 | 11 | 13 | 15 | 17 | 20 | ·02 |
| ·98 | 9550 | 9572 | 9594 | 9616 | 9638 | 9661 | 9683 | 9705 | 9727 | 9750 | 9772 | 2 | 4 | 7 | 9 | 11 | 13 | 16 | 18 | 20 | ·01 |
| ·99 | 9772 | 9795 | 9817 | 9840 | 9863 | 9886 | 9908 | 9931 | 9954 | 9977 | 10000 | 2 | 5 | 7 | 9 | 11 | 14 | 16 | 18 | 20 | ·00 |
| | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | −1 | −2 | −3 | −4 | −5 | −6 | −7 | −8 | −9 | Colog |
| | | | | | | | | | | | | Differences. | | | | | | | | | |

ANTILOGARITHMIC RECIPROCALS.

ANTILOGARITHMIC RECIPROCALS.

## LOGARITHMS 10 TO 99.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 00000 | 04139 | 07918 | 11394 | 14613 | 17609 | 20412 | 23045 | 25527 | 27875 |
| 2 | 30103 | 32222 | 34242 | 36173 | 38021 | 39794 | 41497 | 43136 | 44716 | 46240 |
| 3 | 47712 | 49136 | 50515 | 51851 | 53148 | 54407 | 55630 | 56820 | 57978 | 59106 |
| 4 | 60206 | 61278 | 62325 | 63347 | 64345 | 65321 | 66276 | 67210 | 68124 | 69020 |
| 5 | 69897 | 70757 | 71600 | 72428 | 73239 | 74036 | 74819 | 75587 | 76343 | 77085 |
| 6 | 77815 | 78533 | 79239 | 79934 | 80618 | 81291 | 81954 | 82607 | 83251 | 83885 |
| 7 | 84510 | 85126 | 85733 | 86332 | 86923 | 87506 | 88081 | 88649 | 89209 | 89763 |
| 8 | 90309 | 90848 | 91381 | 91908 | 92428 | 92942 | 93450 | 93952 | 94448 | 94939 |
| 9 | 95424 | 95904 | 96379 | 96848 | 97313 | 97772 | 98227 | 98677 | 99123 | 99564 |

| N | Log N |
|---|---|
| $\pi$ | 0·49715 |
| $180/\pi$ | 1·75812 |
| $e$ | 0·43429 |
| $\mathrm{Log}_{10}e$ | $\bar{1}$·63778 |
| Water d. 62·3 | 1·79449 |
| $g=32\ 2$ | 1·50786 |
| $g=981$ | 2·99167 |
| Speed 88/60 | 0·16633 |
| Ton 2240 | 3·35025 |
| Mile 5280 | 3·72263 |

## LOGARITHMS OF RECIPROCALS.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| ·1 | 100000 | 95861 | 92082 | 88606 | 85387 | 82391 | 79588 | 76955 | 74473 | 72125 |
| ·2 | 69897 | 67778 | 65758 | 63827 | 61979 | 60206 | 58503 | 56864 | 55284 | 53760 |
| ·3 | 52288 | 50864 | 49485 | 48149 | 46852 | 45593 | 44370 | 43180 | 42022 | 40894 |
| ·4 | 39794 | 38722 | 37675 | 36653 | 35655 | 34679 | 33724 | 32790 | 31876 | 30980 |
| ·5 | 30103 | 29243 | 28400 | 27572 | 26761 | 25964 | 25181 | 24413 | 23657 | 22915 |
| ·6 | 22185 | 21467 | 20761 | 20066 | 19382 | 18709 | 18046 | 17393 | 16749 | 16115 |
| ·7 | 15490 | 14874 | 14267 | 13668 | 13077 | 12494 | 11919 | 11351 | 10791 | 10237 |
| ·8 | 09691 | 09152 | 08619 | 08092 | 07572 | 07058 | 06550 | 06048 | 05552 | 05061 |
| ·9 | 04576 | 04096 | 03621 | 03152 | 02687 | 02228 | 01773 | 01323 | 00877 | 00436 |

| N | Log 1/N |
|---|---|
| $\pi$ | $\bar{1}$·50285 |
| $180/\pi$ | $\bar{2}$·24188 |
| $e$ | $\bar{1}$·56571 |
| $\mathrm{Log}_{10}e$ | 0·36222 |
| Water d. 62·3 | $\bar{2}$·20551 |
| $g=32·2$ | $\bar{2}$·49214 |
| $g=981$ | $\bar{3}$·00833 |
| Speed 88/60 | $\bar{1}$·83367 |
| Ton 2240 | $\bar{4}$·64975 |
| Mile 5280 | $\bar{4}$·27737 |

## SUGGESTIONS FOR NOTATION AND PRINTING.

AUTHORS are requested to adhere as far as possible to the suggestions in the following tables of equivalent symbols. The Council of the London Mathematical Society has recently prepared a circular on the question of Mathematical printing, embodying the substance of an earlier circular on the same subject issued by the Royal Society.

The most important of the suggestions are given below. Members of the M.A., who are interested, and who send a stamped addressed envelope to Dr. Bromwich, St. John's College, Cambridge, will receive a copy of the circular in question.

| *Instead of* | *Always print* |
|---|---|
| $\sqrt{2},\ \dfrac{1}{\sqrt{2}},\ \sqrt{13}$ | $\surd 2$ or $2^{\frac{1}{2}}$, $1/\surd 2$ or $2^{-\frac{1}{2}}$, $\surd 13$ or $13^{\frac{1}{2}}$ |
| $\sqrt{ax^2+2bx+c}$ | $\surd(ax^2+2bx+c)$ or $(ax^2+2bx+c)^{\frac{1}{2}}$ |
| $\sqrt{\dfrac{a}{b}}$ | $\surd(a/b)$ or $(a/b)^{\frac{1}{2}}$ |
| $\sqrt{-1}$ | $\iota$ or $i$ |
| $n\,.\,\overline{n+1}\,.\,\overline{n+2}$ | $n(n+1)(n+2)$ |
| $(\underline{\vert n})^2,\ \underline{\vert n+1},\ \underline{\vert 2n},\ 2^n\,\underline{\vert n}$ | $(n!)^2,\ (n+1)!,\ (2n)!,\ 2^n.n!$ |
| $\dot{x},\ \dot{y},\ \dot{r},\ \ddot{\theta}$ | $x',\ y',\ r',\ \theta''$ (by preference) |
| $\dfrac{a}{2},\ \dfrac{a+b}{3},\ \dfrac{a+b+c}{4}$ | $\frac{1}{2}a,\ \frac{1}{3}(a+b),\ \frac{1}{4}(a+b+c)$ |
| $\dfrac{a+b}{c},\ \dfrac{a}{b+c},\ \dfrac{a}{b}+c$ <br> $\dfrac{p}{q},\ \dfrac{p}{q}+\dfrac{r}{s}$ | $(a+b)/c,\ a/(b+c),\ a/b+c$ <br> $p/q,\ p/q+r/s$ (in current text) |
| $p/(q+r/s)$ | $\dfrac{p}{q+r/s}$ or $p\Big/\left(q+\dfrac{r}{s}\right)$ |
| $(p/q+r)/s$ | $\dfrac{p/q+r}{s}$ or $\left(\dfrac{p}{q}+r\right)\Big/s$ |
| $\dfrac{1}{x},\ \dfrac{1}{x^n}$ | $1/x$ or $x^{-1}$, $1/x^n$ or $x^{-n}$ |
| $\dfrac{p+\dfrac{q}{2}}{\dfrac{r}{3}+\dfrac{s}{4}},\ \dfrac{x}{y+\dfrac{z}{t}}$ | $\dfrac{p+\frac{1}{2}q}{\frac{1}{3}r+\frac{1}{4}s},\ \dfrac{x}{y+z/t}$ |
| $e^{-\frac{n\pi x}{l}},\ e^{-\frac{x^2}{4kt}}$ | $e^{-n\pi x/l},\ e^{-x^2/4kt}$ |
| $I_{\frac{\pi}{a}},\ \int_0^{\frac{\pi}{2}},\ \int_0^{\frac{\pi}{n}}$ | $I_{\pi/a},\ \int_0^{\frac{1}{2}\pi},\ \int_0^{\pi/n}$ |
| $(2/l)\int_0^l \sin(r\pi x/l)\sin(s\pi x/l)dx$ | $\dfrac{2}{l}\int_0^l \sin\dfrac{r\pi x}{l}\sin\dfrac{s\pi x}{l}\,dx$ * |

*** It may be noted that an integral with limits always involves a double line: so that the ordinary notation for fraction involves no additional use of "spaces" beyond those required for the integral in any case.**

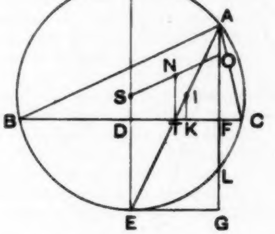
# MATHEMATICAL NOTES.

**452.** [$K^1$. 2. c.] **(vol. viii. No. 117, p. 85, Note 446).** Let $N$ be the centre of the N.P. circle. The rest of the notation as in Mr. Dobbs' note.

$NT$ is $\perp^r$ to $BC$. ($SN$ should meet $AF$ in $O$, the orthocentre.)

As in that note, $\quad DK.KF=r.GL.$

By Euc. II. 13, $$NI^2=NK^2+r^2-2r.NT.$$




Also $(\frac{1}{2}R)^2-NK^2=$ power of $K$ with respect to N.P. circle $=DK.KF=r.GL.$

$$\therefore \quad NI^2=(\tfrac{1}{2}R)^2+r^2-r(2NT+GL).$$

But
$$2NT+GL=SD+OF+GL$$
$$=SD+FL+GL=SE=R;$$
$$\therefore \quad NI^2=(\tfrac{1}{2}R-r)^2. \quad \text{Q.E.D.}$$

E. H. SMART.

**453.** [$K^1$. 20. e.] *Geometrical Proofs of some Triangle Formulae.*

(i) $\Sigma\cos^2 A-2\cos A\cos B\cos C=0.$

If $O$ be the circumcentre, and $E$, $F$ the middle points of $AC$, $AB$, we have

$$OE=R\cos B,\quad OF=R\cos C,\quad E\hat{O}F=180^\circ-A.$$

Hence
$$EF^2=OE^2+OF^2-2OE.OF\cos E\hat{O}F$$
$$=R^2(\cos^2 B+\cos^2 C+2\cos B\cos C\cos A).$$

But
$$EF=\frac{a}{2}=R\sin A. \quad \text{Hence, etc.}$$

(ii) $r=4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}.$

In the figure, $DF$ is the diameter of the circumcircle $\perp^r$ to $BC$, and $DK$, $IL$ are $\perp^r$ to $IC$, $BC$ respectively. Then, since $D\hat{I}C=D\hat{C}I=\dfrac{A}{2}+\dfrac{C}{2}$, we have $DC=DI$.

Hence
$$IC=2CK=2CD\sin\frac{B}{2};$$
$$\therefore \quad r=IC\sin\frac{C}{2}=2CD\sin\frac{B}{2}\sin\frac{C}{2}.$$

But
$$CD=DF\sin\frac{A}{2}=2R\sin\frac{A}{2}. \quad \text{Hence, etc.}$$

(iii) $\sin\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{bc}}.$

In the figure, $BX$, $CY$ are $\perp^r$ to $AI$, and $IM$, $IN$ $\perp^r$ to $AC$, $AB$ respectively. Then, since $BXIN$ is cyclic,

$$\therefore\ B\hat{X}N = B\hat{I}N = 90^\circ - \frac{B}{2},$$

and similarly, $$C\hat{M}Y = C\hat{I}Y = \frac{A}{2} + \frac{C}{2}.$$

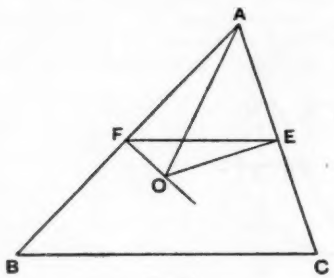



FIG. 1.

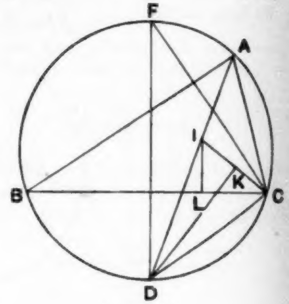



FIG. 2.

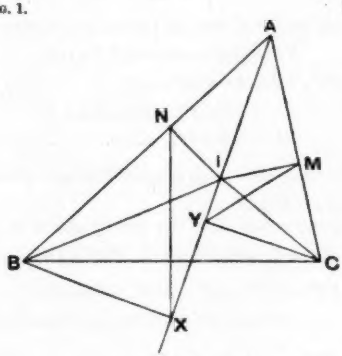



FIG. 3.

Hence $B\hat{X}N = C\hat{M}Y$. Also $A\hat{B}X = A\hat{C}Y$.
Hence $\triangle$s $BXN$, $CMY$ are similar;

$$\therefore\ \frac{BN}{BX} = \frac{CY}{CM};$$

$$\therefore\ BX \,.\, CY = BN \,.\, CM = (s-b)(s-c).$$

But $$\sin\frac{A}{2} = \frac{BX}{c} = \frac{CY}{b};$$

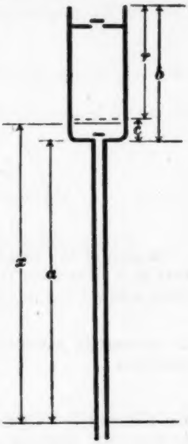
$$\therefore\ \sin^2\frac{A}{2} = \frac{BX \,.\, CY}{bc} = \frac{(s-b)(s-c)}{bc}.$$

E. M. RADFORD.

**454.** [s. 1.] *The Working Conditions of the Common Pump.*

The text-books on Hydrostatics differ in the statement of the working conditions of the common pump. It is the object of the present paper to obtain these conditions in a form free from ambiguity.

Let the length of the pipe, measured vertically from the surface of the water to the bottom of the barrel, be denoted by $a$, the length of the barrel by $b$, the clearance-space (represented as a length) by $c$. Then $b-c$ is the range of the piston; it will be convenient to put $b-c=r$.



Let the height of the water-barometer be denoted by $h$, and the height above the free surface of the level of the water in the pump by $x$.

Initially $x$ is zero. If, as the result of any number of successive strokes of the piston, $x$ never reaches the value $a+c$, then the pump "does not work."

There are two valves, one at the top of the pipe, one in the piston. It is clear that, if during a complete stroke either of these valves does not open, $x$ will be unchanged during the following strokes, and the pump will not work. Consider the action of the piston-valve. This will open during a downstroke if the air below it is sufficiently compressed, and in this case the water-level will rise during the next following upstroke. The pressure below the piston required for the piston-valve to open during the downstroke is just greater than the pressure of the atmosphere.

For the sake of brevity, let us suppose that at the beginning of a downstroke $x>a<a+c$; that is, let us suppose that the water-level is in the clearance-space. The pressure of the air below the piston is then $h-x$ at the beginning of the downstroke, and at the end of the downstroke, if the piston-valve does not open, becomes $(h-x)\dfrac{b+a-x}{c+a-x}$, since $b+a-x$ and $c+a-x$ respectively are the measures of the initial volume and the final volume. The pump will not work unless

$$(h-x)\frac{b+a-x}{c+a-x}>h,$$

i.e. unless

$$h(b-c)>x(a+b-x)$$

or

$$hr>x(a+b-x).$$

The value of $x$ in this inequality may vary from $x=a$ to $x=a+c$. The interpretation of the inequality becomes obvious on plotting the values of $x(a+b-x)$ for values of $x$ from $x=0$ to $x=a+b$. The maximum ordinate of the parabola so obtained is $\frac{1}{4}(a+b)^2$, the corresponding value of $x$ being $\frac{1}{2}(a+b)$.

Three cases present themselves.

(1) $a$ and $a+c$ may both be less than $\frac{1}{2}(a+b)$. In this case the greatest value of $x(a+b-x)$ occurs when $x=a+c$ and is $(a+c)r$. To ensure $hr>x(a+b-x)$ it is only necessary that

$$h>a+c.$$

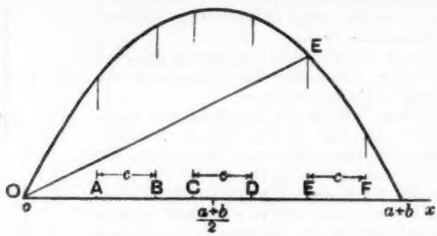
(2) $a$ may be less than $\frac{1}{2}(a+b)$, $a+c$ greater than $\frac{1}{2}(a+b)$. In this case the greatest value of $x(a+b-x)$ occurs when $x=\frac{1}{2}(a+b)$ and is $\frac{1}{4}(a+b)^2$. To ensure $hr>x(a+b-x)$ it is necessary that

$$hr>\tfrac{1}{4}(a+b)^2.$$

(3) $a$ and $a+c$ may both be greater than $\frac{1}{2}(a+b)$. In this case the greatest value of $x(a+b-x)$ occurs when $x=a$ and is $ab$. To ensure $hr>x(a+b-x)$ it is necessary that

$$hr>ab.$$

In the diagram, the first case is illustrated by $OA=a$, $OB=a+c$, the second case by $OC=a$, $OD=a+c$, the third by $OE=a$, $OF=a+c$.




In each of the three cases we have found a single condition. In the first case it is clear from the previous argument, or is easily shown algebraically, that, when $h>a+c$, the relation

$$hr>ab$$

is necessarily satisfied. In the second case we see similarly that the two relations

$$h>a+c,$$
$$hr>ab$$

are necessarily satisfied if $hr>\frac{1}{4}(a+b)^2$. Finally, in the third case, when $hr>ab$, the relation

$$h>a+c$$

is necessarily satisfied.

It may be worth while to notice that

$$hr>ab$$

is the condition that the water shall rise to the top of the pipe. The condition that the water shall rise above a level $x$ in the *pipe* is

$$hr>bx.$$

If in the diagram we introduce ordinates to represent the values of $bx$ from $x=0$ to $x=a$, we obtain a straight line such as, for the third case, $OE'$.

H. E. Schmitz.

**455.** [$P^1$. 3. b. α.] *Retro-Azimuthal Projections* (cf. *Gazette*, No. 108, Dec. 1913, pp. 208 *et sqq.*).

Let $O$ be the centre of the map, and $P$ any point of the map.

It is necessary to distinguish two kinds of R.A. projections.

(i) The bearing of $O$ from $P$ is represented correctly, on the map, by the angle between the straight line $OP$ and the meridian through $P$.

We shall refer to this as of the "First kind."

(ii) The bearing of $O$ from $P$ is represented correctly, on the map, by the angle between the straight line $OP$ and the central meridian.

This will be referred to as of the "Second kind."

The problem has been solved in the following cases:

First kind.—*Orthomorphic.* The Stereographic projection is both Azimuthal and Retro-Azimuthal. This solution is due to Mr. J. I. Craig. It obviously requires no proof.

*The map of the Egyptian Survey Department* belongs to both kinds, seeing that the meridians are straight lines and parallel to one another.

*Equidistant.* A solution by Hammer.

Second kind.—*Orthomorphic.* Solution due to Dr. Maurer, published, on some date in 1911 or 1912, in Petermann. The meridians and parallels are

a system of confocal hyperbolas and ellipses, and the projection is retro-azimuthal with respect to any point on the central meridian.

I owe some of the above information to the courtesy of Mr. Hinks, who was kind enough to lend me a letter from Mr. Craig containing references to the first, third, and fourth of the above cases.

*Retro-Azimuthal Projections of the second kind.*

Taking the earth as a sphere, let $P$ be a point on the sphere (of radius unity), $p$ its correspondent on the plane.

$P$ is in longitude $l$ and latitude $u$. [$l$, of course, is difference of longitude.]

$p$ is the point $(x, y)$.

$O$, the centre of the projection, is in latitude $\alpha$.

The R.A. condition is

$$\frac{y}{x}=\sin u \,.\, \cos l-\cos u \,.\, \tan \alpha \qquad \text{(i)}$$

*Orthomorphic.* Consider the orthomorphic transformation formula

$$y+ix=\sinh(gd^{-1}u+il). \qquad \text{(ii)}$$

This gives

$$\left.\begin{aligned} x&=\sin l \,.\, \sec u,\\ y&=\cos l \,.\, \tan u \end{aligned}\right\} \qquad \text{(iii)}$$

or, transferring the origin to the point $(0, \tan \alpha)$, we have

$$x=\sin l \,.\, \sec u,$$
$$y=\cos l \,.\, \tan u-\tan \alpha.$$

These values of $x$ and $y$ evidently satisfy (i): also, since $\alpha$ is arbitrary the R.A. property is true for every point on $x=0$.

The equations to the meridians and parallels are, obviously,

$$\left.\begin{aligned} \frac{x^2}{\sin^2 l}-\frac{y^2}{\cos^2 l}&=1\\ \text{and}\quad \frac{x^2}{\sec^2 u}-\frac{y^2}{\tan^2 u}&=1 \end{aligned}\right\} \qquad \text{(iv)}$$

referred to the same origin as equations (iii), *i.e.* a point on the equator.

*Equal Area.* In addition to (i), we have*

$$\frac{d'(x, y)}{d'(l, u)}=\cos u. \qquad \text{(v)}$$

Combining these, after some reduction, we get

$$\frac{\cos u \,.\, \cos l+\sin u \,.\, \tan \alpha}{\sin l \,.\, \cos u} \cdot \frac{d'x^2}{d'l}-\frac{\sin u-\cos u \,.\, \cos l \,.\, \tan \alpha}{\sin^2 l \,.\, \cos u} \cdot \frac{d'x^2}{\partial u}=2. \qquad \text{(vi)}$$

Lagrange's equations are

$$\frac{\cos u \,.\, \sin l \,.\, dl}{\cos u \,.\, \cos l+\sin u \,.\, \tan \alpha}=\frac{\cos u \,.\, \sin^2 l \,.\, du}{\sin u-\cos u \,.\, \cos l \,.\, \tan \alpha}=\frac{dx^2}{2}. \qquad \text{(vii)}$$

I have only been able to get a solution to these equations in the special case $\alpha=0$.

In this case the equations reduce to

$$\tan l \,.\, dl=\cot u \,.\, \sin^2 l \,.\, du=dx^2/2.$$

From the first and last of these we have

$$x^2-2 \,.\, \log \sec l=K. \qquad \text{(viii)}$$

Also we have

$$\tfrac{1}{2} \,.\, \cot l \,.\, dx^2=dl,$$
$$\tfrac{1}{2} \,.\, \operatorname{cosec}^2 l dx^2=\cot u \,.\, du;$$

hence

$$\tfrac{1}{2} \,.\, dx^2=\cot u \,.\, du-\cot l \,.\, dl,$$

*i.e.*

$$x^2-2 \,.\, \log(\sin u/\sin l)=K'. \qquad \text{(ix)}$$

* $d'$ stands for $\partial$, *e.g.* $\dfrac{d'(x, y)}{d'(l, u)}$ stands for $\dfrac{\partial(x, y)}{\partial(l, u)}$.

Hence the general solution is

$$x^2-\log\sec^2 l=f[x^2-\log(\sin^2 u/\sin^2 l)]. \qquad \text{(x)}$$

Consider the solution

giving
$$\left.\begin{aligned} x&=\surd(\log\sec l),\\ y&=\surd(\log\sec l)\,.\cot l\,.\sin u.\end{aligned}\right\} \qquad \text{(xi)}$$

Before accepting this solution we must determine the limit, as $l$ approaches zero, of $\surd(\log\sec l)\,.\cot l$.

$$\text{It}=\text{Lt}\cot l\,.\surd[-\log(1-l^2/2)+\ldots]$$
$$=\tfrac{1}{2}, \text{ after an easy reduction.}$$

The equation to a meridian is

$$x=\surd(\log\sec l),$$

and the equation to a parallel is

$$y=x\,.\sin u/\surd(e^{2x^2}-1).$$

G. P. Blake.

Bradfield College, Berks.

**456.** [D. 6. c. δ.] *Investigation of a simple formula for calculating the successive "numbers of Bernoulli."*

Let
$$\frac{x}{e^x-1}=y=a_0+a_1x+a_2\frac{x^2}{\lfloor 2}+\ldots, \qquad (1)$$

where $a_2$, $-a_4$, $a_6$, $-a_8$, ... are Bernoulli's numbers.

Then
$$e^x(y)=y+x;$$
$$\therefore\ e^x(y_1+y)=y_1+1 \text{ and } e^x(y_2+2y_1+y)=y_2, \text{ etc.}$$

$\therefore$ putting $x=0$, $\quad a_1+a_0=a_1+1$ [whence $a_0=1$], $\qquad$ (2)

$$a_2+2a_1+a_0=a_2 \text{ [whence } a_1=-\tfrac{1}{2}], \text{ and so on;}$$

the expansion on the left being of binomial type, with suffixes instead of powers.

If we symbolize $a_2$, $a_3$, $a_4$, ... by $a^2$, $a^3$, $a^4$, etc., and remember that $a_0=1$, the formulae, excluding the first, become

$$(a+1)^2=a^2,\quad (a+1)^3=a^3,\quad (a+1)^4=a^4, \text{ etc.} \qquad (3)$$

Hence we obtain the difference equations,

$$a(a+1)^2=a^2(a-1),\quad a(a+1)^3=a^3(a-1),\quad a(a+1)^4=a^4(a-1), \text{ etc.;} \qquad (4)$$

and, again taking differences,

$$a^2(a+1)^2=a^2(a-1)^2,\quad a^2(a+1)^3=a^3(a-1)^2, \text{ etc.,} \qquad (5)$$

and, generally, $\quad a^r(a+1)^s=a^s(a-1)^r, \qquad (6)$

with the exception of (2) and any processes involving (2).

These are avoided if $s$ is greater than 1.

The series of equations

$$a^2(a+1)^2=a^2(a-1)^2,\quad a^3(a+1)^3=a^3(a-1)^3,\ \ldots\ldots\ a^r(a+1)^r=a^r(a-1)^r,$$

shows that all the odd $a$'s, except $a_1$, are zero; and the equations

$$a(a+1)^2=a^2(a-1),\quad a^2(a+1)^3=a^3(a-1)^2,\ \ldots\ldots,$$
$$i.e.\quad a^n(a+1)^{n+1}=a^{n+1}(a-1)^n,$$

determine in succession the values of $a_2$, $a_4$, ... $a_{2n}$.

Thus, from $a(a+1)^2=a^2(a-1)$ we have $a_3+2a_2+a_1=a_3-a_2$;

$$\therefore\ 3a_2=-a_1;\quad \therefore\ a_2=\tfrac{1}{6};$$

and, from the next equation, we obtain

$$a_5+3a_4+3a_3+a_2=a_5-2a_4+a_3, \text{ also } a_3=0;$$
$$\therefore\ 5a_4+a_2=0;\quad \therefore\ a_4=-\tfrac{1}{30}.$$

When $n>1$, we can use the general equation,

$$a^n(a+1)^{n+1}=a^{n+1}(a-1)^n, \text{ and omit the odd } a\text{'s};$$

whence, writing $C_r$ for ${}^nC_r$ and $C_r'$ for ${}^{n+1}C_r$, we have

$$C_1'a_{2n}+C_3'a_{n-2}+C_5'a_{n-4}+\ldots+C_1a_{2n}+C_3a_{n-2}+C_5a_{n-4}+\ldots=0\,;\ldots\ldots\ldots(7)$$

*e.g.* if $n=2, 3, 4, 5, 6$ in succession,

$$(3+2)a_4+(1+0)a_2=0, \qquad \therefore\ a_{11}=-\tfrac{1}{30};$$

$$(4+3)a_6+(4+1)a_4=0, \qquad \therefore\ a_6=\tfrac{5}{7} \text{ of } \tfrac{1}{30}=\tfrac{1}{42};$$

$$(5+4)a_8+(10+4)a_6+(1+0)a_4=0, \qquad \therefore\ a_8=\tfrac{1}{9}\{-\tfrac{1}{3}+\tfrac{1}{30}\}=-\tfrac{1}{30};$$

$$(6+5)a_{10}+(20+10)a_8+(6+1)a_6=0, \qquad \therefore\ a_{10}=\tfrac{1}{11}\{1-\tfrac{1}{6}\}=\tfrac{5}{66};$$

$$(7+6)a_{12}+(35+20)a_{10}+(21+6)a_8+(1+0)a_6=0,$$

$$\therefore\ a_{12}=\tfrac{1}{13}\{-\tfrac{25}{6}+\tfrac{9}{10}-\tfrac{1}{42}\}=-\tfrac{691}{2730}.$$

The positive values of these fractions, beginning with $a_2$, are Bernoulli's numbers, generally denoted by $B_1, B_2, \ldots$. The above results give them as far as $B_6$. The next one, $B_7$, is $\frac{7}{6}$; after that they are heavy numbers.

The advantage of formula (7), as a basis of calculation, over the formula (3), which is usually taken as the basis, is two-fold:

(1) The calculation of any coefficient $a_{2n}$, when $n$ is even, depends on $\frac{1}{2}n$ previous coefficients instead of $n+1$ coefficients; and, when $n$ is odd, it depends on $\frac{1}{2}(n-1)$ coefficients instead of $n+1$.

(2) The multipliers needed in (7) are much lighter than those of (3), as they consist of pairs of binomial coefficients of degrees $n+1$ and $n$ instead of binomial coefficients of degree $2n+1$.

For example, $a_{18}$ is calculated from 4 terms, with binomial multipliers of the $9^{\text{th}}$ and $10^{\text{th}}$ degrees, instead of from 10 terms with binomial multipliers of the $19^{\text{th}}$ degree.

If desired, formula (7) may be written in the form

$$(2n+1)a_{2n}+\frac{2n-1}{3}C_2\,.\,a_{2n-2}+\frac{2n-3}{5}C_4\,.\,a_{2n-4}+\ldots,$$

but it is rather easier to work in its original form, and it is probably easiest to *remember* in the unreduced symbolic form,

$$a^n(a+1)^{n+1}=a^{n+1}(a-1)^n.$$

A. LODGE.

**457.** [L[1]. a.] In Russell's *Pure Geometry* (1893, p. 47) it is "proved" that a pair of points is a conic by reciprocating a pair of straight lines. Now a pair of straight lines, $AOB$, $COD$, is a conic only when considered as the limit of a hyperbola, *i.e.* if the order of description be $AOCDOBA$ or $AODCOBA$. The tangent to such a conic rotates at $O$, and the true reciprocal is seen to be either the finite straight line which is the limit of an ellipse, or the straight line with a finite gap which is the limit of a hyperbola. The paradox that two points, though a conic, cannot be obtained as a section of a cone, is thus avoided.

A similar argument applies to the statement in Askwith's *Analytical Geometry* (1908, p. 397), that the tangential equation of the second degree represents two points when it reduces to the form

$$(Al+Bm+Cn)(Al'+B'm+C'n)=0.$$

C. W. ADAMS.

**458.** [D. 6. c.] *On certain coefficients connected with the expansions of* $(e^x-1)^n$, $(xD)^nf(x)$, *and* $(x+1)(x+2)(x+3)\ldots(x+n)$.

Let $f, f_1, f_2, \ldots$ denote any function of $x$ and its $1^{\text{st}}, 2^{\text{nd}}, \ldots$ differential coefficients.

By operating two or three times, it is seen that $(xD)^n f$ may be expressed in the form

$$k_{n,1}xf_1+\frac{k_{n,2}}{|\underline{2}}x^2f_2+\ldots+\frac{k_{n,n}}{|\underline{n}}x^nf_n\,;$$

where the $k$'s are independent of $x$, and of the form of $f(x)$; also

$$k_{n,1}=1,\quad k_{n,n}=|\underline{n}.$$

It will be found that

$$k_{n,r}=r^n-r(r-1)^n+\frac{r(r-1)}{|\underline{2}}(r-2)^n-\ldots+(-1)^{r+1}\cdot r,$$

which is the coefficient of $\dfrac{x^n}{|\underline{n}}$ in $(e^x-1)^r$.

For the expansion of this last see Boole's Finite Differences, and for the formula,

$$k_{n,r}=r\{k_{n-1,r-1}+k_{n-1,r}\}\,;$$

and by using this, a table of the $k$'s may easily be formed.

Some of the summations may be effected without much trouble; we have

$$\frac{1}{|\underline{r}}k_{r+2,r}=\frac{r(r+1)(r+2)}{|\underline{4}}(3r+1),$$

whence

$$r^{r+1}-(r-1)(r-1)^{r+1}+\frac{(r-1)(r-2)}{|\underline{2}}(r-2)^{r+1}-\ldots=\frac{|\underline{r+2}}{|\underline{4}}(3r+1).$$

Similarly
$$\frac{1}{|\underline{r}}k_{r+3,r}=\frac{r(r+1)(r+2)(r+3)}{|\underline{6}}(15r^2+15r),$$

and
$$r^{r+2}-(r-1)(r-1)^{r+2}+\frac{(r-1)(r-2)}{|\underline{2}}(r-2)^{r+2}-\ldots=\frac{|\underline{r+3}}{|\underline{6}}(15r^2+15r).$$

To connect these coefficients with the expansion of

$$(x+1)(x+2)(x+3)\ldots(x+n),$$

let the latter be written in descending powers of $x$ in the form

$$\alpha_n x^n+\beta_n x^{n-1}+\gamma_n x^{n-2}+\ldots,$$

and let us write $\dfrac{k_{n,n}}{|\underline{n}}=a_n,\quad \dfrac{k_{n,n-1}}{|\underline{n-1}}=b_n,\quad \dfrac{k_{n,n-2}}{|\underline{n-2}}=c_n,$ etc.;

we shall find that if $\beta_n=\phi(n)$, then $b_n=\phi(-n)$;
if $\gamma_n=\chi(n)$, then $c_n=\chi(-\overline{n-1})$;
if $\delta_n=\psi(n)$, then $d_n=\psi(-\overline{n-2})$;

the same law of formation continuing; here $b$ and $\beta$ may be interchanged, also $c$ and $\gamma$, etc. As negative indices are inadmissible, the $k$'s must be summed (see above) before using these results.

These quantities form recurring series, continued to infinity in either direction; consider for example the $c$'s and $\gamma$'s. The series for these is

$$\ldots\ 65,\ 25,\ 7,\ 1,\ 0,\ 0,\ 0,\ 2,\ 11,\ 35,\ 85,\ \ldots$$

(the scale of relation being

$$u_n-5u_{n-1}+10u_{n-2}-10u_{n-3}+5u_{n-4}-u_{n-5}=0).$$

This series may be denoted by either $c$'s or $\gamma$'s, admitting negative suffixes in each case.

Thus $\gamma_{-5}=65,\quad \gamma_{-4}=25,\quad \gamma_{-3}=7,$ etc.;

employing $c$'s, we have $c_6=65,\quad c_5=25,\quad c_4=7,$ etc.

Bernoulli's numbers may be expressed in terms of the $k$'s in several ways.

Let $\dfrac{x}{e^x-1}=B_0+B_1x+\dfrac{B_2}{\lfloor 2}x^2+\dfrac{B_3}{\lfloor 3}x^3+\ldots$ $(B_{2n+1}=0$, except $B_1)$;

then we have

(i) $B_n=-\dfrac{1}{2}+\dfrac{1}{3}k_{n,2}-\dfrac{1}{4}k_{n,3}+\ldots$ to $n$ terms; (see Boole)

(ii) $-\dfrac{2(2^{n+1}-1)}{n+1}B_{n+1}=-\dfrac{1}{2}+\dfrac{1}{2^2}k_{n,2}-\dfrac{1}{2^3}k_{n,3}+\ldots$ to $n$ terms;

(iii) $B_{2n-2}=\dfrac{4}{2n+1}\left\{1-\dfrac{1}{2^3}k_{2n,2}+\dfrac{1}{3^3}k_{2n,3}-\ldots-\dfrac{1}{(2n)^3}k_{2n,2n}\right\}$;

(iv) $\dfrac{2^n(2^n-1)}{n}B_n=\sum\limits_{r=1}^{r=n}(-1)^{r+1}\cdot\dfrac{1}{r}\dfrac{\cos\frac{r\pi}{4}}{2^{\frac{r}{2}-1}}k_{n,r}$, $(n>1)$.

G. Osborn.

**459.** [D. 6. b; V. a. $\theta$.] It is usual to make the proofs of the theorems

$$\lim_{x\to 0}\frac{\sin x}{x}=1=\lim_{x\to 0}\frac{\tan x}{x}$$

depend on certain geometrical inequalities, commonly taken as intuitive, which involve the length of a circular arc. It does not seem to be generally recognised that the radian can be introduced in a natural way, the above theorems and others connected with them proved, and the attribution of length to a circular arc justified without further recourse to geometry than is necessary for the proof of the addition theorems. The following notes indicate the method:

1. If $s_n$ be the sum of the first $n$ terms of a sequence steadily increasing (decreasing) in the stricter sense, then the sequence of which the $n$th term is $\dfrac{s_n}{n}$ steadily increases (decreases) in the stricter sense.

2. So long as all the multiples concerned of the angle specified by $a$ are acute angles, the sequences of which the $n$th terms are

$$\sin na-\sin(n-1)a,\quad \tan na-\tan(n-1)a$$

steadily decrease, increase respectively in the stricter sense. For

$$\sin na-\sin(n-1)a=2\sin\tfrac{1}{2}a\,.\cos\tfrac{1}{2}(2n-1)a,$$
$$\tan na-\tan(n-1)a=\sin a\sec na\sec(n-1)a.$$

3. With the same restriction, the sequences of which the $n$th terms are $\dfrac{\sin na}{n}$, $\dfrac{\tan na}{n}$ steadily decrease, increase respectively in the stricter sense. From (1, 2).

4. Extension of (3) in an obvious way to the case where $n$ specifies any positive number satisfying the condition of (2) leads to the theorems: $\dfrac{\sin na}{n}$, $\dfrac{\tan na}{n}$ are respectively steadily decreasing, increasing functions of $n$ so long as $na$ specifies an acute angle; and hence obviously to the theorems: $p\sin\dfrac{1}{p}\,.\,a$, $p\tan\dfrac{1}{p}\,.\,a$ are respectively steadily increasing, decreasing functions of $p$ so long as $p$ specifies a positive number and $\dfrac{1}{p}\,.\,a$ an acute angle.

5. Since, further, $p\tan\dfrac{1}{p}\,.\,a-p\sin\dfrac{1}{p}\,.\,a=\dfrac{1}{2p^2}\,.\left(p\tan\dfrac{1}{p}\,.\,a\right).\left(2p\sin\dfrac{1}{2p}\,.\,a\right)^2$ is positive and tends to zero as $p\to\infty$, therefore

$$\lim_{p\to\infty}p\sin\frac{1}{p}\,.\,a,\quad \lim_{p\to\infty}p\tan\frac{1}{p}\,.\,a$$

exist and are equal. For this theorem there is no necessary restriction on $a$.

6. $\lim\limits_{p\to\infty} p\sin\frac{1}{p}.\theta$ is directly proportional to $\theta$. For if

$$\lim_{p\to\infty} p\sin\frac{1}{p}.\,a=a,\quad \lim_{p\to\infty} p\sin\frac{1}{p}.\,\theta=x,$$

and $\theta : a = m$, then

$$x=\lim p\sin\frac{1}{p}.\,\theta = m\lim\left(\frac{p}{m}.\sin\frac{p}{m}.\,a\right)=ma.$$

Hence the unit of angle may be so chosen that the measure of the (variable) angle specified by $\theta$ is $\lim\limits_{p\to\infty} p\sin\frac{1}{p}.\,\theta$. In this system the measure of a straight angle is denoted by $\pi$.

7. Discarding now the expression "sine (etc.) of an angle" in favour of "sine (etc.) of the number which is the measure of the angle," in terms of the unit adopted in (6), the theorems proved may be stated: if $x$ be any positive number, then, as $p\to\infty$, $p\sin x/p$, $p\tan x/p$ steadily increase, decrease respectively toward $x$ as limit. The theorems

$$\lim_{x\to 0}\frac{\sin x}{x}=1=\lim_{x\to 0}\frac{\tan x}{x}$$

follow immediately, as also the theorems:

$$\frac{\sin x}{x},\quad \frac{\tan x}{x}$$

steadily decrease, increase respectively as $x$ increases from 0 to $\pi/2$.

8. Let $a$ be a (fixed) positive number, $a_1, a_2, \dots a_n$ positive numbers such that (i) $\sum\limits_{r=1}^{n} a_r = a$; (ii) $\lim\limits_{n\to\infty} a_r = 0$ $(r=1, 2, \dots n)$, then

$$\lim_{n\to\infty}\sum_{r=1}^{n}\sin a_r = a = \lim_{n\to\infty}\sum_{r=1}^{n}\tan a_r.$$

For $$\Sigma\sin a_r = a - \Sigma a_r\left(1-\frac{\sin a_r}{a_r}\right),\ \text{etc.}$$

Similarly for tan. This justifies the attribution of length to an arc of a circle.

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**460.** [K[1]. 6. a; V. a. ι.] *A Note on the Teaching of Co-ordinate Geometry.*

There are certain treatments of mathematical problems which we may perhaps term qualitative methods. To one such method I should like to call attention. It will often be found useful in a preliminary examination of a curve whose equation is given, and will serve for a quick rough location of a drawing with regard to the two sides of some particular line.

There is always a difficulty in deciding which is to be taken as the positive and which the negative side of a line whose equation is given. It is easy enough to ascertain whether two points whose co-ordinates are given lie on the same or on opposite sides of a given line; but any convention as to positive-or negative-ness always seems arbitrary.

With the conventional method of drawing the axes of co-ordinates, i.e. with the axis of $x$ always drawn from left to right and the axis of $y$ always from the foot of the page towards the top—directly or obliquely—there is always a right-hand and a left-hand side to any straight line—always an upper and a lower side; and it is quite easy to show that, if the point $(x', y')$ is off the line $Ax+By+C=0$, it lies to the right or left of the line according as $\dfrac{Ax'+By'+C}{A}$ is positive or negative, and above or below the line according as $\dfrac{Ax'+By'+C}{B}$ is positive or negative.

In fact, if $P$ is the point $(x', y')$ and $PM$, $PN$ parallel to $Ox$, $Oy$ meet the line in $M$ and $N$ respectively, it is an excellent elementary exercise to show that $MP = \dfrac{Ax'+By'+C}{A}$ and that $NP = \dfrac{Ax'+By'+C}{B}$.

This may be used consistently throughout a scheme of teaching co-ordinate geometry in making preliminary investigations regarding any conic whose equation is given. Such a fact as that the lines

$$Ax+By+C \pm \lambda(A'x+B'y+C')=0$$

are harmonically conjugate with regard to

$$Ax+By+C=0 \text{ and } A'x+B'y+C=0^*$$

is rendered easily demonstrable.

The parabola $(\alpha x+\beta y)^2+2gx+2fy+c=0$ can be written

$$\left(\frac{\alpha x+\beta y}{\alpha}\right)^2 = -\frac{2g}{\alpha^2} \cdot \frac{2gx+2fy+c}{2g}.$$

The left-hand side of the equation being always positive, so must be the right-hand side; and this shows that the whole curve lies to the right or left of the line $2gx+2fy+c=0$ (which it touches, where $\alpha x+\beta y=0$ cuts it) according as $g$ is negative or positive.

If the general equation of the second degree,

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

represents an ellipse. Then, writing it in the form

$$x^2+\frac{2h}{a}xy+\frac{b}{a}y^2 = -\frac{2g}{a} \cdot \frac{2gx+2fy+c}{2g},$$

since $ab-h^2$ is positive the left-hand side of the equation is always positive; and again we have that the whole curve lies entirely to the right or left of the straight line $2gx+2fy+c=0$, according as $\frac{g}{a}$ is negative or positive.

If again the equation represents an hyperbola, it is an easy algebraical exercise to transform it to the form

$$(a_1x+b_1y+c_1)(a_2x+b_2y+c_2)+K=0;$$

and, writing this $\quad \dfrac{a_1x+b_1y+c_1}{a_1} \cdot \dfrac{a_2x+b_2y+c_2}{a_2} = -\dfrac{K}{a_1a_2},$

we see that if $\dfrac{K}{a_1a_2}$ is negative, the expressions

$$\frac{a_1x+b_1y+c_1}{a_1}, \quad \frac{a_2x+b_2y+c_2}{a_2}$$

must be of the same sign; *i.e.* the whole curve must lie to the right or to the left of *both* the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_1=0$; but that if $\dfrac{K}{a_1a_2}$ is positive, it must lie to the right of one and to the left of the other of these lines, or *vice versa*.

This idea may be extended to solid Geometry: for, with the conventional way of drawing a representation of the planes of reference, there is always a right-hand and a left-hand side to any other plane, always an upper side and a lower side, always a front and a back: and it can be seen that the point $(x', y', z')$ is to the right or left of the plane $Ax+By+Cz+D=0$, according as $\dfrac{Ax'+By'+Cz'+D}{A}$ is positive or negative, in front of or behind the plane, according as $\dfrac{Ax'+By'+Cz'+D}{B}$ is positive or negative; above or below the plane, according as $\dfrac{Ax'+By'+Cz'+D}{C}$ is positive or negative.

R. W. K. EDWARDS.

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* And hence that, for all values of $\lambda$, $Ax+By+C\pm\lambda(A'x+B'y+C')=0$, are conjugate rays of a pencil in involution.

**461.** [C. 2. j; L[1]. 9. c.] *Approximate Formulae for the Perimeter of an Ellipse.*

The perimeter ($P$) is $4aE(\theta)$, where $E(\theta)$ is obtained from tables of elliptic functions, and $\theta$ is arc sin $e$.

Also the perimeter is given by the series $2\pi a(1-\frac{1}{4}e^2-\frac{3}{64}e^4-\frac{5}{256}e^6-\dots)$ which converges too slowly for rapid calculation.

The unsatisfactory formula $2\pi a(1-0{\cdot}364e^2)$, found in a text book, led to an investigation into the availability of an expression of this type, or of that commonly used: $2\pi a\sqrt{1-0{\cdot}5e^2}$. If such a type is used, it can only be approximate for a limited range of values of $b/a$, and two formulae are really necessary; even then it is difficult to obtain two which cover the whole range and give satisfactory results for the middle cases. The formulae suggested for this type are no. (ii) below. A logarithmic formula, however, gives good results except for values of $b/a$ less than 0·15, which would rarely occur. For this I am indebted to Professor J. P. Dalton, of Johannesburg.

The following values for $P/2\pi a$ may be used within the range specified, with an error of less than one per cent.

| | Formula for $P/2\pi a$. | Range for $b/a$. |
|---|---|---|
| (i) | $2E(\theta)/\pi$ | 0 – 1 |
| (ii) | $1-0{\cdot}28e^2$ | 0·64 – 1 |
| | $1-0{\cdot}21e^2-0{\cdot}13e^4$ | 0·25 – 1 |
| (iii) | $1+\log_{10}(0{\cdot}998-0{\cdot}55e^2)$ | 0·15 – 1 |
| (iv) | $\sqrt{(1-0{\cdot}5e^2)}$ | 0·7 – 1 |
| (v) | $\frac{1}{2}(1+b/a)$ | 0·7 – 1 |
| (vi) | $\sqrt{(b/a)}$ | 0·8 – 1 |

The following are values of $1000P/2\pi a$, calculated by the different formulae:

| $b/a$ | 1 | 0·985 | 0·940 | 0·866 | 0·766 | 0·707 | 0·643 | 0·5 | 0·342 | 0·174 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (i) | 1000 | 992 | 970 | 934 | 887 | 861 | 831 | 771 | 712 | 662 | 637 |
| (ii) | 1000 | 992 | 967 | 930 | 885 | 860 | 836 | | | | |
| | 1000 | 994 | 974 | 939 | 891 | 862 | 833 | 770 | 714 | 675 | 660 |
| (iii) | 999 | 992 | 970 | 935 | 887 | 859 | 829 | 768 | 710 | 667 | 651 |
| (iv) | 1000 | 992 | 970 | 935 | 891 | 866 | 841 | 791 | 747 | 718 | 707 |
| (v) | 1000 | 993 | 970 | 933 | 883 | 854 | | | | | |
| (vi) | 1000 | 992 | 970 | 931 | 875 | | | | | | |

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## REVIEWS.

**Plane Geometry,** Parts I. and II. By G. St. L. Carson and D. E. Smith. Pp. 480. Each Part, 2s. 6d. (Ginn.)

The publication of Part II. completes this work, of which the earlier portion has already been noticed in the *Gazette*. In their preface the authors express the hope "that the spirit of scholarship and the sense of aesthetic value in geometry, as set forth in this work, may appeal to a large number of English-speaking teachers of mathematics." The authors have aimed high; an examination of their work will show that they have not failed. Their book is one which Euclid, if he could get acquainted with it, would probably read with interest and approval. He might regret that the old maxim, "no superfluity," was not held in such regard as was usual in his times;

but he would not fail to see that the subject was clearly developed, that the explanations shirked no difficulties, and that the propositions were fully demonstrated. He would recognise the honour done to the ancients by the way in which Book V., which deals with similar triangles and polygons, leads up to one of their great achievements, the construction of a polygon equivalent to one given polygon and similar to another, and he would appreciate the Pythagorean climax of Book VI. (on regular polygons), in the construction of the regular pentagon. In the appendix he would find theorems on maxima and minima, too little known to this day in English schools, and two final problems on the sides of a regular polygon which will enable the modern schoolboy to understand how it was that Archimedes was able to get a close approximation to the value of $\pi$.

Book IV. deals with ratio and proportion, and the use of these in the geometry of the rectangle and the triangle. The treatment of ratio is exact, and the subject is well explained. The ratio of two magnitudes $A$ and $B$ of the same kind is said to be that of $m$ to $n$, $A$ being made of $m$ parts, and $B$ of $n$ parts, all the parts being equal to one another. Thus to deal with a ratio such as that of $A$ to $B$ it is necessary to find a numerical ratio which is its equivalent. The difficulty of the method of sub-multiples (which involves the finding of a common measure) is avoided by the use of the theorem that if $nA=mB$, $A:B$ must be equal to $m:n$. This leads to an explanation of a method by which, using multiples, the ratio $m:n$ can be found where it is possible to do so. The ratio $m:n$ must be, in such a case, a rational ratio. Thus we get part of Euclid's definition of proportion in a new form, "If the magnitudes of two ratios are commensurable, the two ratios are equal if both ratios are equal to the same rational ratio." A similar transformation of the remainder of Euclid's definition supplies a means of defining the equality of two ratios when the magnitudes of one of the ratios are incommensurable. From this point the proofs of the fundamental propositions in proportion are set out in two parts, and the authors suggest that beginners should in each case learn only the first of these parts, that is, the part dealing with commensurable magnitudes. The only numerical ratios used are rational ratios.

The subject of Book III. is the geometry of the circle. Its thirty propositions contain rather more of the subject than Euclid put into his book of the same number. The tangent at a point $P$ on a curve is defined as a line drawn from $P$ such that chords of unlimitedly short length, also drawn from $P$, make angles of unlimited smallness with the line. A postulate has to be used, that "an unlimitedly short line drawn from any point subtends an unlimitedly small angle at any other point." If $P$ is the first point, $O$ the second, and $PQ$ the diminishing line, the postulate is true however near $O$ is to $P$. The beginner may be reminded that in such a case, as in all others, the line $OP$ remains of fixed length while the diminution of $PQ$ is in progress.

Another point may be mentioned. In the earlier parts of this book the authors were careful to give complete demonstrations of locus-theorems. In dealing with the theorem (in Book IV.) that the locus of a point which moves so that its distances from two fixed points are in a given ratio is a circle they show that—$A$ and $B$ being the fixed points, $CD$ the diameter of the circle, $A$, $C$, $B$, $D$ being in a straight line—any point at which $AC$, $CB$ subtend equal angles lies on the circle, but they do not show that $AC$, $CB$ subtend equal angles at any point on the circle. The omission may be due to the fact that the point has been described as moving. There is here a concealed postulate that the motion is possible.

There is no doubt that this book would bring a permanent benefit to any mind that had intelligently studied it. It has been said that appreciation of an ordered whole of instruction such as this is only possible to an adult mind. But there are some minds which are sufficiently developed at fairly early ages to feel its value; as for the others, one can only say that the introduction and the various permissions and explanations scattered through the book make the study of geometry as easy as it is possible to make it; no book can make the study a really easy one. This book can be confidently recommended to any teacher who is in search of a text-book for his pupils.

T. M. A. Cooper.

**A Theory of Time and Space.** By ALFRED A. ROBB, M.A. Pp. vi + 373. 10s. 6d. net. 1914. (Cambridge University Press.)

**The Principles of Relativity.** By E. CUNNINGHAM, M.A. Pp. xiv + 222. 9s. net. 1914. (Cambridge University Press.)

**The Theory of Relativity.** By L. SILBERSTEIN, Ph.D. Pp. viii + 295. 10s. net. 1914. (London: Macmillan & Co.)

Clerk Maxwell, in his remarkable manual of Elementary Science called *Matter and Motion*, which was published in 1876 by the Society for Promoting Christian Knowledge, closes a short paragraph on Absolute Space with these words:

"All our knowledge, both of time and space, is essentially relative. When a man has acquired the habit of putting words together, without troubling himself to form the thoughts which ought to correspond to them, it is easy for him to frame an antithesis between this relative knowledge and a so-called absolute knowledge, and to point out our ignorance of the absolute position of a point as an instance of the limitation of our faculties. Any one, however, who will try to imagine the state of a mind conscious of knowing the absolute position of a point will ever after be content with our relative knowledge."

The section preceding that just quoted is "On the Idea of Time," and in it Maxwell speaks of two events of astronomical interest, of which the first named must have occurred either before or after the other, or else at the same time. Mr. Robb, however, in his book on *Space and Time*, does not admit that an event which is neither before nor after another event is necessarily simultaneous with it. According to the view brought forward by him the only events which are really simultaneous are events which occur at the same place. When events occur at different places, all that can be said of them is that one is neither before nor after the other. The meaning is illustrated by a reference to Fizeau's experiment for determining the velocity of light. Let a flash be sent out from a point $P$ at an instant $A$, and be reflected from $Q$ so as to come back to $P$ at the instant $C$. Then the instant $B$ at which it reaches $Q$ must be after $A$ and before $C$; but we have no means of identifying any particular instant between $A$ and $C$ at the point $P$ with the instant $B$ at the point $Q$. So long as we keep to the set of instants of which any one individual is directly conscious, or the set of instants which a single particle of matter occupies, one instant which is neither before nor after another is identical with it; but this cannot be logically asserted of sets of instants occupied by different particles. Mr. Robb is thus led to the idea of what he calls "conical order"; from which conception with the aid of twenty-one postulates and more than two hundred theorems he constructs a geometry in which every element is determined by four coordinates. These coordinates may be made to correspond to the usual space coordinates $x$, $y$, $z$, and the time coordinate $t$.

In this four dimensional space a point represents a state of a particle at a given time; and from Mr. Robb's point of view "the theory of space becomes absorbed in the theory of time, spacial relations being regarded as the manifestation of the fact that the elements of time form a system in conical order—a conception which may be analysed in terms of the relations of After and Before." This Theory of Space and Time is in short a novel presentation of those inter-relations which are usually referred to as coming within the domain of the Theory of Relativity. Mr. Robb is meanwhile content with having established a consistent geometry of space and time relationships, in which a greater precision of definition of certain somewhat elusive kinematical conceptions is given than has hitherto been possible. Further developments and applications are left for a future volume.

On the other hand, Mr. Cunningham and Dr. Silberstein aim at giving, in their respective books, a complete account of the present position of the Theory of Relativity. Both trace its historic development through the works of Maxwell, Hertz, Heaviside, Larmor, Lorentz, Einstein, and Minkowski. Both cover, to a large extent, the same ground; and both are to be congratulated on the way in which they have accomplished their task. Each author has thoroughly assimilated in his own mind the whole doctrine of relativity, and gives his own characteristic presentation of it.

Broadly speaking, Mr. Cunningham and Dr. Silberstein discuss in similar fashion the course of events, both experimental and theoretical, which led Lorentz to the transformation formulae known by his name, and Einstein to the enunciation of the underlying principles of relativity. Thereafter, however, especially in their treatment of Minkowski's conceptions, the two authors part company. Mr. Cunningham follows Minkowski in the use of the four dimensional calculus with the four-vector and six-vector. Dr. Silberstein works with the quaternion calculus throughout.

In his first paper on the electromagnetic equations, Minkowski referred to the possibility of using Hamilton's quaternion calculus instead of Cayley's matrix calculus, but regarded the former as too narrow and clumsy. This is not the opinion of Dr. Silberstein ; nor is it the view of Professor Conway, who was the first to show (*Proc. R.I.A.* 1911) how to apply Hamilton's calculus to the relativity principle. It is interesting to note that the non-quaternionic vector analyses do not lend themselves at all readily to this kind of investigation, although, as Mr. Cunningham's pages abundantly testify, any form of vector analysis notation is of great service in all electromagnetic theory. The broad distinction between Minkowski's matrix method and the quaternion treatment advocated by Conway and Silberstein is clearly put in the following quotation from the latter :

"In advocating here the cause of quaternions I am doing so not only because they furnish us very short formulae and simplify their handling. Quite independently of this the quaternion seems to me intrinsically better adapted than the world-vector to express that 'union' of time and space which was (too strongly, perhaps) emphasized by Minkowski. For although there is a certain union between the two, which manifests itself when we pass from one system to another, there is no total fusion. In each system, out of the four scalars $x$, $y$, $z$, $l$, the first three are more intimately bound to one another than any of them to the last one. The first three are artificial components of a vector, $\mathbf{r}$, which certainly is a more immediate entity than each of them. Now, in a four-vector, as well as in a matrix, $x$, $y$, $z$, $l$ are, as it were, on entirely equal footing with one another, being the four 'components' of the former, or the 'four' constituents of the latter. On the other hand, a quaternion $q$ has a distinct vector part $Vq$, and a scalar part $Sq$, and none of the components of the former can be confounded with the latter. Now, the position of a particle is determined by a vector (in its ordinary sense), and its date by a scalar. What then more natural than to take the first as the $V$ and to embody the second in the $S$ of the quaternion ? . . . Let us therefore combine the position vector $\mathbf{r}$ of a particle with its date $l=\iota ct$ [where $\iota=\sqrt{-1}$], into a quaternion

$$q=l+\mathbf{r},$$

which, if it needed a name of its own, we might call the position quaternion . . . an abbreviation for 'position-date quaternion.'"

Thus, as originally pointed out by Conway, the quaternion takes the place of Minkowski's four-vector ; and the six-vector is immediately symbolised by Hamilton's bi-vector, namely, $\mathbf{r}+\iota\mathbf{s}$. Then there are the closely connected quaternion differential operators

$$\nabla-\iota c^{-1}\frac{\partial}{\partial t}, \quad \nabla+\iota c^{-1}\frac{\partial}{\partial t},$$

in which $c$ is the speed of light, and of which the successive application gives rise to the operator

$$\nabla^2+c^2\frac{\partial^2}{\partial t^2}.$$

Minkowski's *lor* and *lor'* are practically these differential operators, whose laws of combination had already been studied by quaternionists.

Different though the methods of treatment are, as indicated by the account just given, the same problems are discussed by both authors, such as longitudinal and transversal mass, electromagnetic mass, the application of relativity principles to dynamics as well as to electrodynamics, momentum and energy in the aether, the structure and kinetic properties of the electron,

the question of an objective aether, and the like. As regards the question of an objective aether, Mr. Cunningham develops his theory that a mechanical moving aether is consistent with the principle of relativity, provided that the total velocity of the aether at any point is equal to that of light. Neither author has, however, anything fundamental to say concerning gravitation, which, as Mr. Cunningham puts it in his last chapter, refuses to fit in with our slight knowledge of the constitution of matter. Mr. Cunningham's general conclusion is that, although the principle of relativity is irrelevant to much of our experimental knowledge, yet it is consistent with it all, and gives beside a new unity to our thought.

It would be invidious to say which of the two authors has the more perfectly attained his purpose. Each will, no doubt, appeal to his own audience. Meanwhile, we cannot but congratulate ourselves that we now possess in the English language two admirable treatises on the theory of Relativity. As for Mr. Robb's *Theory of Space and Time* it is altogether unique. There is not its like in any other language.

C. G. Knott.

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